

# Do Instantons and Strings Cluster when the Number of Colors is Large?

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## Abstract

We consider the  $N_c \rightarrow \infty$  limit of QCD using a toy model in which instantons exchange color-singlet scalar fields which do not self-interact. Our main observation is that collective attraction leads the formation of large clusters containing  $O(N_c)$  nonperturbative objects. We further show that this clustering of instantons is limited due to a non-trivial repulsion inherent in the ADHM multi-instanton solution. As a result the vacuum is very different from that at low  $N_c$ , notably being more inhomogeneous, in ways which will affect chiral symmetry breaking of light quarks. We also briefly discuss a similar phenomenon for color strings in baryons made of medium-mass (charm-like) quarks.

When localized in sufficiently large numbers, even weakly interacting bosons can bind together. Although, for example, electroweak interactions between  $W$ ,  $Z$ , and Higgs bosons are of order  $\alpha_w \sim 1/100$ , a hundred or more may collectively bind and form a sphaleron. In this paper we consider similar phenomena with nonperturbative QCD objects, concentrating on instantons but eventually turning to strings as well.

In QCD, the instanton liquid model [1, 2] has been consistently successful in describing the physics of light hadrons (see Ref. [3] for a review). Of central importance is its description of phenomena related to chiral symmetry breaking, confirmed in detail by recent lattice studies of the lowest Dirac

eigenvalues [4, 5]. The primary parameters of this model are the number density of instantons (plus anti-instantons) and their average size, determined long ago from phenomenology to be  $n \approx 1 \text{ fm}^{-4}$  and  $\bar{\rho} \approx 1/3 \text{ fm}$ , respectively [1]. With these numbers one obtains an outstanding quantitative description of hadronic correlation functions; for example, the most accurately known vector and axial correlators (from  $\tau$  decays) are reproduced at all distances literally within the (rather small) experimental error bars [6].

At the same time, this picture of the QCD vacuum remains incomplete. The instanton ensemble does not explain confinement effects. Furthermore, the apparent suppression of large-size instantons remains unexplained dynamically<sup>1</sup>.

Other open questions are related to the behavior of the instanton ensemble in the large  $N_c$  limit. Long ago, Witten emphasized that small-size instantons should be *exponentially* suppressed [8] but, on the other hand, he recently found that the topological susceptibility in large  $N_c$  gluodynamics is merely *power* suppressed [9], as  $\chi \sim \Lambda^4 O(N_c^0)$ . In general, there are many possible scenarios in which instantons of size  $\rho \sim 1/\Lambda$  can satisfy both conditions. The first steps towards a numerical resolution of this issue were recently made in Ref. [10], in which instantons were studied for  $N_c = 2 - 5$ . Although these authors found a suppression of small-sized instantons, those of greater sizes are enhanced, suggesting that the total topological susceptibility is finite at large  $N_c$ .

In this letter we begin by noticing that in the  $N_c = 3$  instanton vacuum the gauge fields are distributed inhomogeneously such that the field strength is concentrated in small regions of space-time, the diluteness parameter being  $n\bar{\rho}^4 \sim (1/3)^4$ . The gluonic correlators consequently decrease much more rapidly with distance than those of mesons containing light quarks, giving the “glueball” spectrum a mass scale larger than that of the light mesons.

As  $N_c \rightarrow \infty$ , we will suggest that the instanton ensemble becomes even *more inhomogeneous* than the two and three color cases studied to date. This idea apparently contradicts the earlier view that in the large- $N_c$  limit the so-called “master field” must be simple and homogeneous [11]. And yet this picture does agree with recent progress regarding the  $N_c \rightarrow \infty$  limit of  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SUSY YM) theory. In that

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<sup>1</sup> For a recent attempt to relate the small size of instantons with a small radius of QCD strings see [7].

case the master field is an *instanton cluster* in which all instantons share a common location and size [12]. This is in perfect agreement with Maldacena’s conjecture of the AdS/CFT correspondence [13], where the Anti-deSitter 5-d space is manifested simply as the instanton measure of collective coordinates,  $d^4z d\rho/\rho^5$ .

Despite the very profound differences between  $\mathcal{N} = 4$  SUSY YM theory and QCD-like theories – the former is a conformal theory without dimensional parameters, and clustering results from the exchange of adjoint fermions not present in QCD – we argue below that this phenomenon is in fact rather generic. Indeed, similar clustering of instantons should also occur in QCD at large  $N_c$ .

Our primary argument for this conjecture is based on the idea that, although they do not interact via direct color forces, gluonic objects of completely different colors are generically attracted through effective interactions. We will specifically describe instantons exchanging a colorless scalar field. Although the corresponding coupling becomes weak at large  $N_c$ , this weakness is overcome by the large ( $O(N_c^2)$ ) number of pairs and clustering ensues<sup>2</sup>.

The toy model we use couples the gauge fields to a color-singlet scalar field, which can be interpreted as the  $0^{++}$  glueball. This scalar naturally interacts with *any* gluonic configurations – instantons, flux tubes, etc. Not only is this the lightest dynamical state in pure glue theory, it is also has been found to be extraordinarily small, with a size of about 0.2 fm [14]. For this reason it may be reasonable to consider it as an elementary dynamical field, analogous to pion-mediated dynamics in nuclei. The action we have in mind is an Euclidean one with the simplest form encoding our degrees of freedom:

$$S = \int d^4x \left[ \frac{1}{4g^2} G_{\mu\nu}^a(x) G_{\mu\nu}^a(x) + \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \lambda \phi(x) G_{\mu\nu}^a(x) G_{\mu\nu}^a(x) \right]. \quad (1)$$

Here the gauge field strength tensor is  $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$ , the parameter  $\lambda$  is a coupling constant with the dimension of length, and as usual the factor  $g$  has been absorbed in the gauge field. While one could naturally include a potential for the field  $\phi$  we see no motivation to do so in this case.

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<sup>2</sup> Note that this attraction is similar to that between nucleons at large distances. However, unlike instantons or strings, they are fermions and as such have a strong repulsive core which leads to a saturation density in their clusters (nuclei).

Minimizing Eq. (1) generates the equations of motion:

$$\left(\frac{1}{4g^2} + \lambda\phi(x)\right) D_\mu G_{\mu\nu}^a(x) = 0, \quad (2)$$

$$\partial^2\phi(x) - \lambda G_{\mu\nu}^a(x) G_{\mu\nu}^a(x) = 0. \quad (3)$$

The self-dual instanton solution, unchanged by the presence of  $\phi$ , is

$$A_\mu^a(x) = \frac{2\eta_{\mu\nu}^a x_\nu}{(x-z)^2 + \rho^2}, \quad (4)$$

where  $\eta_{\mu\nu}^a$  is the usual 't Hooft symbol, with  $\rho$  and  $z$  the instanton size and position.

The parameter  $\lambda$  can be determined by the scalar glueball mass,  $m_\phi \simeq 1.5$  GeV. After squaring the interaction term and Fourier transforming the non-local result, the leading contribution to the glueball self-energy is

$$\Pi(q^2) = (32\pi^2)^2 \lambda^2 \int d\rho \nu(\rho) (q\rho)^4 K_2(q\rho)^2, \quad (5)$$

where  $K_2(x)$  is a modified Bessel function. From the static limit the mass is obtained,

$$m_\phi^2 = \Pi(0) = (32\pi^2)^2 \lambda^2 n. \quad (6)$$

With this one finds  $\lambda = 0.024$  fm in order to fit a glueball mass of 1.5 GeV. The glueball mass is assumed to be stable with respect to the number of colors [15], whereas the instanton density is assumed to scale as  $n \sim N_c$  according to Ref. [2]. Thus we must have a decreasing coupling constant

$$\lambda = \frac{\lambda_0}{\sqrt{N_c}}, \quad (7)$$

where  $\lambda_0$  may be fixed by the previous case of  $N_c = 3$ .

The attraction between any pair of pseudo-particles is quantified by the change in the action, parametrized as usual by the instanton sizes and the separation between their centers,  $R$ . For two isolated instantons, this is

$$\Delta S_{pair}(R, \rho_1, \rho_2) = -\frac{\lambda^2}{2} \int d^4x d^4y [G_{\mu\nu}(x - R, \rho_1)]^2 D(x - y) [G_{\mu\nu}(y, \rho_1)]^2. \quad (8)$$

where the scalar propagator in Euclidean space is

$$D(x) = \frac{m}{4\pi^2|x|} K_1(m|x|), \quad (9)$$

and the sources are the field strength tensor evaluated over an instanton action distribution. This will describe the extent of interactions between two instantons in completely distinct SU(2) color subgroups, as well as the interaction between an instanton and anti-instanton.

According to Eqs. (7) and (8), the attractive force between any pair of pseudo-particles will fall as  $1/N_c$  due to the reduced coupling constant. Yet this force remains relevant, and in fact becomes dominant, when one considers a cluster of instantons with a number of constituents on the order of the number of colors. Specifically, the total change in the action from such a cluster is parametrically large:

$$\Delta S_{clust} = \sum_{i>j} \Delta S(R, \rho_i, \rho_j) \sim N_c^2 \overline{\Delta S_{pair}} \sim N_c. \quad (10)$$

Thus it will overcome the entropy of random placements, so that the large- $N_c$  ensemble will be one of strongly bound instanton clusters with  $\sim N_c$  constituents. This force will be strongest between instantons which share a common size, and in this way the size distribution will be sharply narrowed.

Can such a clustering process continue indefinitely, until all of them collapse into the same mega-cluster? The answer is negative for two reasons. The first is that instantons and anti-instantons in the same SU(2) subgroup would obviously annihilate one another. The second, less trivial, effect which would limit clustering is the rather interesting non-linear deformation of the two-instanton solution. Although this is, strictly speaking, only derived for instanton pairs in the same color subgroup, we think it will generally lead to a kind of shell model situation (albeit without fermions) in which instantons and anti-instantons tend to occupy only *non-overlapping* SU(2) subgroups. We now consider both effects in detail.

Any group of pseudo-particles will likely include as many anti-instantons as instantons, for the scalar exchange knows no difference<sup>3</sup>. The inevitable

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<sup>3</sup> Consequentially, in this ensemble the instantons and anti-instantons of different colors are statistically independent and therefore its topological susceptibility is *not suppressed* relative to an unclustered ensemble,  $\chi \sim N_c \Lambda^4$ . A more refined model with more complicated interactions is therefore needed to provide for Witten's suppression,  $\chi \sim \Lambda^4$ , such as one that restricts the topological charge of each cluster to  $Q = O(1)$ .

annihilation between instantons and anti-instantons within a given cluster will leave some fraction of the original number in a cluster of instantons or anti-instantons alone<sup>4</sup>. Statistical fluctuations in  $N_I - N_{\bar{I}}$  will however guarantee that these clusters persist, with a number of constituents still on the order of  $N_c$ .

Instantons in the same  $SU(2)$  subgroup may be analyzed using an exact multi-instanton solutions known as the ADHM construction [16]. We find that this leads to an effective repulsion for two reasons.

First, when two such pseudo-particles converge, their distinct peaks in the action density are deformed continuously into a toroidal cloud in two of the four spatial directions [17]. There is a minimal separation,  $R_{min} = \sqrt{\rho_1 \rho_2 / 2}$ , after which a certain coordinate transformation reveals re-separation into a direction orthogonal to the original  $R_\mu$ . Although the classical action remains constant, this modifies the non-trivial measure of the two-instanton coordinate space at the quantum level (in the pre-exponent of the partition function). So although the manifold is smooth, we deduce that a coordinate discontinuity must exist to ensure that the same configuration is not counted twice (an observation, to the best of our knowledge, not made in literature before). This minimal separation is an effective hard-core repulsion between instantons of the same color projection.

The second effective repulsion arises from interactions between different color subgroups. Note that the action density becomes less concentrated as two instantons in the same subgroup are deformed into a toroidal-like configuration. Consider now the interaction between this toroidal configuration and a separate cluster of instantons, each of a different adjoint color and hence having unmodified shapes. The following calculation reveals that the total attraction in this case is less than that of a cluster interacting with two color-independent, unmodified instantons. Indeed, instead of Eq. (8) we now would have

$$\Delta S_{pair}(R, \rho_1, \rho_2) = -\frac{\lambda^2}{2} \int d^4x d^4y [G_{\mu\nu}(x - R, \rho_1)]^2 D(x - y) \frac{1}{2} s(y, \rho_2, \rho_2), \quad (11)$$

where

$$\frac{1}{4} s(x, \rho_1, \rho_2) = -\frac{1}{2} \partial^2 \partial^2 \ln \det [\Delta(x)^\dagger \Delta(x)] \quad (12)$$

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<sup>4</sup> In the calculations which follow all statements about instanton clusters naturally apply to those made of anti-instantons.

is the action density for the ADHM solution.

The matrix  $\Delta(x)$  is a rather complicated function and we refer the interested reader to Ref. [17] for a thorough analysis. Since we are only interested in quantitative scalar exchanges between an instanton and such a solution, we consider two instantons in the same  $SU(2)$  subgroup and with a color group angle of  $\pi/2$  between the two (the case of maximal deformation). In that case, we have

$$\begin{aligned} \det [\Delta(x)^\dagger \Delta(x)] &= \left[ (x - R)^2 + \rho_1^2 + \frac{(\rho_1 \rho_2)^2}{R^2} \right] \\ &\times \left[ x^2 + \rho_2^2 + \frac{(\rho_1 \rho_2)^2}{R^2} \right] - \frac{(2\rho_1 \rho_2)^2}{R^2} x_1^2. \end{aligned} \quad (13)$$

Here we have chosen  $R_\mu = (R, 0, 0, 0)$  and color group elements  $\sigma_0$  and  $\sigma_1$  for the two pseudo-particles, which leads to a torus in the plane of coordinates  $x_0$  and  $x_1$ .

With this specification we have computed via Monte-Carlo the “binding action” generated by scalar exchange between an instanton and half an ADHM pair, which is compared to that for two more prosaic instantons in Fig. 1. From this we can see that the cluster formation will prefer filling each  $SU(2)$  subgroup once to allow for maximal attraction. This generates a medium of isolated clusters with the order of  $N_c$  individual instantons.

We now depart from pure glue theory and consider the effects of light fermions on the clustering process. Specifically, we discuss QCD-like fermions with fundamental color charges rather than the gluino-like adjoint fermions present in supersymmetric theories<sup>5</sup>.

The clustering of instantons as described effectively reduces the instanton density,  $n$ , by a factor of  $N_c$ . This will have severe consequences for the spontaneous breaking of chiral symmetry. In a background of unclustered instantons, the chiral condensate and quark effective mass scale as [2]:

$$\langle \bar{\psi} \psi \rangle \sim N_c, \quad M_q \sim N_c^0. \quad (14)$$

In a clustered environment we will instead have

$$\langle \bar{\psi} \psi \rangle \sim \sqrt{N_c}, \quad M_q \sim \frac{1}{\sqrt{N_c}}. \quad (15)$$

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<sup>5</sup> The difference is crucial, since the former will have  $O(N_f)$  and the latter  $O(N_c)$  fermionic zero modes.

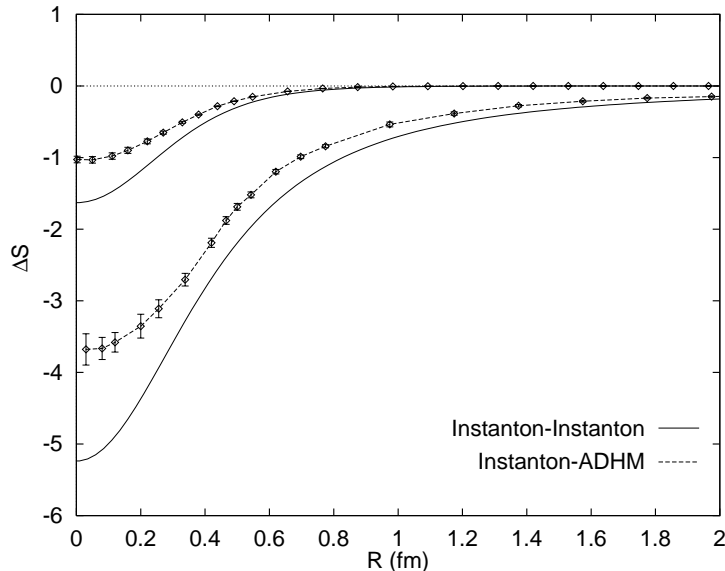


Figure 1: Interaction strength of scalar exchange between either an instanton-instanton pair or instanton-ADHM pair as a function of separation, for  $m_\phi = 1500$  MeV (upper lines) and  $m_\phi = 0$  (lower lines) ( $\rho_1 = \rho_2 = 1/3$  fm for all).

Spontaneous chiral symmetry breaking is thus much stronger with unclustered instantons, and a large number of quark flavors might keep the system in a phase of strongly broken chiral symmetry rather than the clustered medium. To estimate how many flavors would be necessary, we note that the fermion determinant built of zero-mode overlap matrices would raise the action, possibly overcoming the reduction from scalar exchange (8) which will appear in the exponent of the partition function. For clustering to persist we must have

$$\det(\mathcal{T})^{2N_f} e^{-N_c \Delta S_{pair}} \equiv e^{-\Delta S_{eff}} > 1, \quad (16)$$

where  $\mathcal{T}$  is the matrix of overlaps. To extract the  $N_c$  dependence from the determinant, we note that the overlap matrices involve two zero-mode propagators and hence

$$\ln \det(\mathcal{T}) \sim \ln \left( \frac{R^3}{R_0^3} \right) \sim \frac{3}{4} \ln \left( \frac{n_0}{n} \right) \sim \frac{3}{4} \ln N_c. \quad (17)$$



The total effective change in the action is thus

$$\Delta S_{eff} = \frac{3}{2} N_f \ln N_c + N_c \Delta S_{pair} . \quad (18)$$

This becomes positive – and clustering becomes unstable – for

$$\begin{aligned} N_f &> -\frac{2}{3} \Delta S_{pair} \frac{N_c}{\ln N_c} \\ &\gtrsim \frac{2N_c}{\ln N_c} , \end{aligned} \quad (19)$$

where a typical value for  $\Delta S_{pair}$  has been inserted. Thus we see that the clustering phenomena will be inhibited by a rather small  $N_f/N_c$  value.

As already mentioned, the arguments presented above would likely apply not only to instantons but to any configuration of color fields. Color flux tubes (QCD strings) in particular are highly localized color field distributions, in two rather than four dimensions. Assuming flux tubes of different colors can also interact via colorless glueball exchange, the strings will cluster in the large- $N_c$  limit in a similar fashion. Profound consequences for hadronic structure would follow.

Consider for example a baryon made of medium-mass (charm-like) quarks<sup>6</sup>. It has  $N_c$  flux tubes originating from each quark and ending in a central “junction”. The standard picture would be that the quark motion is described by a 3-d string potential,  $V(x) \sim |\vec{x}|$ , around the junction. However if the flux tubes spontaneously cluster along one direction, as we suggest, quark motion will be constrained to that direction and the potential becomes one dimensional. If so, the large classical deformation of such baryons leads to large momentum of inertia and an unusually soft rotational band.

In addition to the colorless attractive forces described here in our toy model, clusters could also form due to confinement effects, if confinement is indeed due to a dual superconductor and its universal U(1) dual Higgs mechanism [18]. Because the Higgs VEV must vanish at the center of *any* topological object, including both instantons [7] and flux tubes, it will be energetically favorable to have these objects at a minimal number of locations.

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<sup>6</sup> Hadrons made of light quarks have different features and baryons made of them are skyrmion-like, whereas those made of very heavy ( $b$ -like) quarks are dominated by perturbative Coulomb forces. Only charmonium spectroscopy is described well by a linear potential and flux tubes.

The Higgs field has been shown numerically to be rather robust for small  $N_c$ , restricting the color flux tubes to small radii (and perhaps instantons to small sizes [7]), and so it is easy to imagine its effects playing a major role in the large- $N_c$  limit.

How might our predictions be tested? Perhaps most directly by lattice simulations. Color groups of currently-simulated SU(4) [19] and higher contain orthogonal SU(2) subgroups, allowing for computation of the correlation strengths between instantons and/or color strings. The number of colors could also be increased to computational limits, and if, as we expect, the nonperturbative objects cluster, many observables would show very different qualitative behavior. Lattice calculations along these lines would provide the most reliable test of vacuum configurations in the large- $N_c$  limit.

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